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Numerical estimates of local dimension by waiting time and quantitative recurrence

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Abstract

Waiting time indicators are defined by measuring the time needed for a point x to approach a given point y. Our aim is to test the use of waiting time indicators, and quantitative recurrence ones to numerically estimate the *local dimension* of attractors in dynamical systems, as suggested by some rigorous results.

We test this approach on a collection of examples, whose dimensions are already known rigorously, or with a large amount of precision. The main outcome is that the use of waiting time, compared to recurrence increases largely the speed of convergence of the estimators.

Keywords: Computational methods in statistical physics; Attractor; Dimension; Quantitative recurrence

1. Introduction

The fractal dimension of attractors is one of the main features of chaotic dynamics. It is well known that its numerical estimate is not an easy task; this is true both in the case of time series arising from experimental data or when we numerically simulate the behaviour of some model.

We emphasise two main difficulties when computing dimensions: the need of many experimental points and the need of high precision computations. Numerical errors can destroy the small scale structure of fractals, which, however can be studied only when a large number of points are determined. On the other hand the theoretical (rigorous) computation of attractors dimension can be performed only in relatively few examples (mainly conformal maps or uniformly hyperbolic attractors [1]) which are not physically relevant. For this reason many different notions of dimension are proposed in literature and many techniques, to estimate them, are studied. In this work we will approach the study of the dimension by considering the use of *quantitative recurrence* and *waiting times*.

We assume the system to be equipped with some physical invariant measure μ. Such a measure "contains" all the physically relevant statistical features of the system (see Ref. [2] for a survey about physical measures).

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Using this approach, the fractal properties of the system are described by the notion of *dimension of the measure*; let the Hausdorff dimension of a set B be denoted by $H(B)$, then the Hausdorff dimension of the measure μ is defined as the dimension of the smallest set having full measure: $d_H(\mu) = \inf_{\mu(B)=1}(H(B))$. This is a global notion related to the dimension of attractors where μ concentrates. One can consider also a *local* notion of the *dimension* of μ. Let us consider the ball $B(x,r)$ centred at x, with radius $r>0$, we say that the local dimension of μ at x is $d_\mu(x)$, if $\mu(B(x,r))\sim r^{d_\mu(x)}$, when $r \to 0$.[1]

This notion of local dimension can be related to other available notions of dimension. Concerning the relations between local and global dimension of a measure, it is known (see Ref. [1]), that if the local dimension exists (i.e., the limit in footnote 1 exists) and it has constant value d almost everywhere, then $d = d_H(\mu)$. Such systems are called *exact dimensional*. For example, systems with non-zero Lyapunov exponents almost everywhere, are exact dimensional [3]. Moreover, local dimension received particular interest, because of the systems with non-uniform fractal behaviour (multifractals), for which the local dimension depends on the point x.

The knowledge of the properties of the physical measure μ, implies the knowledge of the fractal dimension of attractors. It is clear, however, that the knowledge of μ is not an easy task in general. It is as difficult as to compute the dimension of the attractors. A way to handle this problem is to use *recurrence times*. This is based on the following heuristic idea: an estimate of the measure of $B(x,r)$ is obtained by computing the number of iterates needed for the point x to come back in the ball $B(x,r)$. According to Kac's Theorem, in an ergodic system the average time needed for a point, starting from $B(x,r)$, to come back in $B(x,r)$ is $\mu(B(x,r))^{-1}$ (here the average is made over all the points of the ball). Thus taking smaller and smaller radii r, once we know $\mu(B(x,r))$ we can estimate the local dimension. This kind of approach has been already used in physical literature (see for example, Refs. [4–6]) to get estimates of dimension of some strange attractors.

However, we point out that the use of Kac's Theorem for these purposes, must be considered carefully, in fact in the ball $B(x,r)$ there could be a relatively large subset of points having return time smaller than the average and a small set of points having return time much larger than the average (this happens, for example, in the case of rotations by a Liouville number, see Refs. [7,8]). In this case, taking at random, a point in $B(x,r)$ we have a large probability to select a point with too short return time. In some cases we can be faced to other kinds of problem, for instance when using recurrence to estimate multifractal spectra [8,9].

In mathematical literature, a precise analysis of recurrence and its relations with other features of dynamic has been done by defining, in several ways, *quantitative recurrence indicators*. These are defined by measuring the *first return time* of an orbit in a decreasing sequence of neighbourhoods of the starting point.[2] A natural generalisation of the quantitative recurrence indicators can be defined by measuring how fast the orbit starting at some point x, approaches to *another* point y. Indicators of this type have been called *waiting* (*or hitting*) *times* (precise definitions are given below).[3]

In computer simulations or experimental situations, these indicators can be easily estimated by iterating some orbit until it enters in a given neighbourhoods and hence, this construction provides an indicator that seems to be suited for numerical estimations.

In this work, we will consider sequences of balls as neighbourhood of the point we approach. The relations obtained in Refs. [7,11,12] (see also Theorem 1) between quantitative recurrence, waiting times and local dimension can be used as rigorous theoretical support for the numerical estimate of the latter.

Let us now introduce some notations and recall some known results. We will consider a discrete time dynamical system (X,T), where X is a separable metric space equipped with some Borel locally finite measure μ and $T : X \to X$ is a measurable map. Let us fix two points x and y in X, and let us consider the first hitting time of the orbit of x with the ball, $B(y,r)$, centred at y and radius r:

$$\tau_r(x,y) = \min\{n \in \mathbf{N} : T^n(x) \in B(y,r)\}.$$

[1]More precisely, we require that $\lim_{r\to 0}(\log \mu(B(x,r)))/\log r = d_\mu(x)$. In general, we denote $\limsup_{r\to 0}(\log \mu(B(x,r)))/\log r = \overline{d}_\mu(x)$ and $\liminf_{r\to 0}(\log \mu(B(x,r)))/\log r = \underline{d}_\mu(x)$.

[2]For instance, cylinders generated iterating some partition or balls. These indicators can be related to other important features of dynamics: dimension, entropy, orbit complexity, Lyapunov exponents, mixing properties [8].

[3]Once again waiting times with respect to cylinders and balls can be related to other features of dynamics (see Refs. [7,8,10] and references therein).

When we consider the behaviour of $\tau_r(x,y)$, for fixed x, y and $r \searrow 0$, in many interesting cases one can find a power law r^{-R}, as for the local dimension (see footnote 1). Thus one can define the waiting time indicators

$$\overline{R}(x,y) = \limsup_{r\to 0} \frac{\log(\tau_r(x,y))}{-\log(r)}, \quad \underline{R}(x,y) = \liminf_{r\to 0} \frac{\log(\tau_r(x,y))}{-\log(r)}.$$

The indicators $\overline{R}(x)$ and $\underline{R}(x)$ of quantitative recurrence defined in Ref. [12] are obtained as a special case, $\overline{R}(x) = \overline{R}(x,x)$, $\underline{R}(x) = \underline{R}(x,x)$.

For general systems, the quantitative recurrence indicator gives a *lower* bound on the dimension [11,12], whereas the waiting time indicator gives an *upper* bound [7] of the local dimension of the measure at the point y. More precisely one has

Theorem 1. *If X is a closed subset of $\mathbf{R}^n$, μ is an invariant measure, then for almost every $x \in X$:*

$$\overline{R}(x,x) \leqslant \overline{d}_\mu(x) \quad \textit{and} \quad \underline{R}(x,x) \leqslant \underline{d}_\mu(x).$$

Moreover, for any y

$$\underline{R}(x,y) \geqslant \underline{d}_\mu(y) \quad \textit{and} \quad \overline{R}(x,y) \geqslant \overline{d}_\mu(y) \tag{1}$$

holds for μ-almost every x.

These results can be combined to have upper and lower bounds on the (upper and lower) local dimension of general systems, thus we conclude that recurrence and waiting time can provide a *rigorous method* to numerically estimate the local dimension in dynamical systems.

Moreover, it can be proved that if some technical assumptions are satisfied, the recurrence or waiting time indicators are a.e. equal to the dimension of the invariant measure. For example, this is the case for uniformly hyperbolic attractors or fast mixing systems (see Refs. [12–14] for recurrence and Refs. [7,8] for waiting time). In other words, for this kind of systems both upper ad lower estimators converges to dimension.

On the other hand, there are systems for which the recurrence (and waiting) time indicators do not converge to the dimension. Such systems share some particular arithmetic or statistic anomaly. An example [12,15] is provided by the rotation of the unit circle $x \mapsto x + \alpha \pmod 1$ by a *well-approximable* irrational, for example, α to be a Liouville number. This kind of phenomenon is, however, generally believed to be rare and non-generic.

When performing our numerical experiments, we observe that a main problem for the use of recurrence to estimate local dimension, is the quite slow rate of convergence to dimension, and moreover, the existence of large fluctuations. This problem can be already found in the literature and several tricks have been used to circumvent it. These tricks are usually based on performing averages on several points and are not suited to the computation of local dimension for a single point.

In this paper, we consider several dynamical systems whose attractors have local dimension rigorously known or for which good independent estimates are available in literature, thus these values can be used to make a comparison with the ones obtained using our methods. More precisely, we will choose a typical point on the attractor, we estimate the local dimension both by recurrence and waiting time, and then we compare these results with the local dimension obtained by some theoretical argument when available, or using some numerical independent methods, i.e., box counting.

The first outcome of our experiments is that the use of *waiting times*, compared to quantitative recurrence *reduces fluctuations* and *improve convergence speed*. This is due to the possibility of performing averages over different starting points (even to estimate the local dimension at a *single* point). In fact, we remark that (1) holds for many x. A second remark, is that the use of waiting times allows us to investigate the behaviour of the local dimension even at non-typical points, in fact (1) holds for every point y.

We performed our numerical experiments, using Maple software, this because of the easy way in which Maple handles hundred precision digits when necessary (see Refs. [16,17] about the use of Maple for the simulation of dynamical systems and for other experiments about recurrence in dynamical systems). The maple scripts we used in this work can be found at http://www2.ing.unipi.it/~d80288/maplewait/.

2. Numerical experiments

In this section, we describe the numerical simulations we did and our results. For several discrete time dynamical systems we give an estimate of the local dimension at some (randomly) chosen point y on the attractor using the previously introduced indicators in the following way.

(1) *Recurrence*: we choose a point y on the attractor at which we want to estimate the local dimension. We fix a sequence of radii $r_i = 1/i^2$, thus to estimate the recurrence time we compute $\tau_{r_i}(y, y)$ by iterating the point y until it enters the ball $B(y, r_i)$. Then from the plot of $(\log(\tau_{r_i}(y, y)))/ - \log(r_i)$ against $\log(r_i)$, we can extract a measure for the quantitative recurrence at y.

(2) *Waiting times*: we fix again a point y on the attractor where we want to compute the local dimension. Then we choose a finite, but large (usually we choose a number between 10 and 200, depending on the dynamical system we are considering, see next sections for more details), number of random test points $x_1, \ldots, x_n$, which do not need to be on the attractor, then for each point x_k we compute $\tau_{r_i}(x_k, y)$ by iterating x_k until it enters the ball $B(y, r_i)$. Finally we compute the averaged value: $\langle R_{r_i}(y)\rangle = (1/n)\sum_{k=1}^{n}(\log(\tau_{r_i}(x_k, y)))/ - \log(r_i)$. A measure of the waiting time at y is thus obtained plotting $\langle R_{r_i}(y)\rangle$ versus $\log(r_i)$.

In the next sections we show in detail the previous procedure applied to several well-known discrete time dynamical systems.

2.1. Skinny Baker map and perturbed skinny Baker map

The metric space X is the square $[-1, 1] \times [-1, 1]$ and the *Baker map* is defined by

$$T^{bkr}(x, y) = \begin{cases} (\lambda_1 x + c_1, 2y - 1) & \text{if } y \geqslant 0, \\ (\lambda_2 x + c_2, 2y + 1) & \text{if } y < 0, \end{cases}$$

where $0 < |\lambda_1|, |\lambda_2| < 1$ and $(c_i)_{i=1,2}$ are real parameters, such that

$$|\lambda_i + c_i| \leqslant 1 \text{ and } | - \lambda_i + c_i| \leqslant 1, \quad i = 1, 2. \tag{2}$$

It is well known that the attractor is a Cantor set whose Hausdorff dimension (see e.g. Ref. [1]) is $1 + s$, where s the unique root of the equation $|\lambda_1|^s + |\lambda_2|^s = 1$.

Strictly speaking the classical Baker map is for parameters values: $\lambda_1 = \lambda_2 = \frac{1}{2}$ and $c_1 = -c_2 = \frac{1}{2}$. According to [18] if $|\lambda_1 + \lambda_2| < 1$, the map is called *skinny*, otherwise it is *fat*.

A more interesting map obtained modifying the Baker one, is the *slanted Baker map*, it acts on the square $[-1, 1] \times [-1, 1]$ by

$$T_0^{sl\,bkr}(x, y) = \begin{cases} (\lambda_1 x + \mu_1 y + c_1, 2y - 1) & \text{if } y \geqslant 0, \\ (\lambda_2 x + \mu_2 y + c_2, 2y + 1) & \text{if } y < 0, \end{cases}$$

where $(\lambda_i)_{i=1,2}$, $(\mu_i)_{i=1,2}$, $(c_i)_{i=1,2}$ are real parameters which together with (2) satisfy

$$|\lambda_i + \mu_i + c_i| \leqslant 1 \quad \text{and} \quad | - \lambda_i + \mu_i + c_i| \leqslant 1, \ \ i = 1, 2. \tag{3}$$

Simon [19] (see also Ref. [1]) proved that if $\mu_1 \neq \mu_2$ and $0 < |\lambda_1|, |\lambda_2| < |\mu_1 - \mu_2|$, then the Hausdorff dimension is still $1 + s$, where s is the unique root of $|\lambda_1|^s + |\lambda_2|^s = 1$. Moreover, this is true for any small enough perturbation on the first component. We thus considered a perturbed version of this map by adding a small non-linear term of the form $\varepsilon(x^2 - 1)$:

$$T_\varepsilon^{sl\,bkr}(x, y) = \begin{cases} (\lambda_1 x + \varepsilon(x^2 - 1) + \mu_1 y + c_1, 2y - 1) & \text{if } y \geqslant 0, \\ (\lambda_2 x + \varepsilon(x^2 - 1) + \mu_2 y + c_2, 2y + 1) & \text{if } y < 0, \end{cases}$$

where the perturbation is such that the square $[-1, 1] \times [-1, 1]$ is preserved and ε gives the strength of the non-linearity.

We thus compared the local dimension at a random point on the attractor predicted by the Simon result, with an estimate of the local dimension obtained using recurrence and waiting times. The results are presented

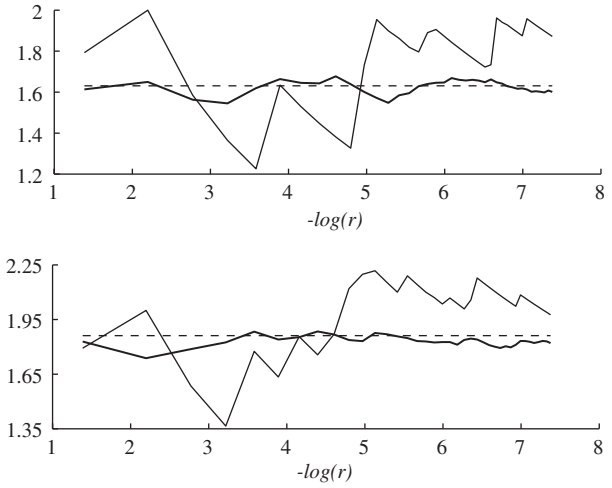

Fig. 1. Baker map. On the top parameters are: $\lambda_1 = \lambda_2 = 1/3$, $c_1 = 0.51$ and $c_2 = -0.512211122$, whereas on the bottom $\lambda_1 = \lambda_2 = \frac{1}{\sqrt{5}}$, $c_1 = 0.51$ and $c_2 = -0.512211122$. In both figures thin line is $-\log(\tau_r(y, y))/\log(r)$, thick line $\langle R_r(y)\rangle$ with a mean over 50 different initial points (x_k), and horizontal dashed line represents the theoretical dimension $1 + s$ obtained from Ref. [19].

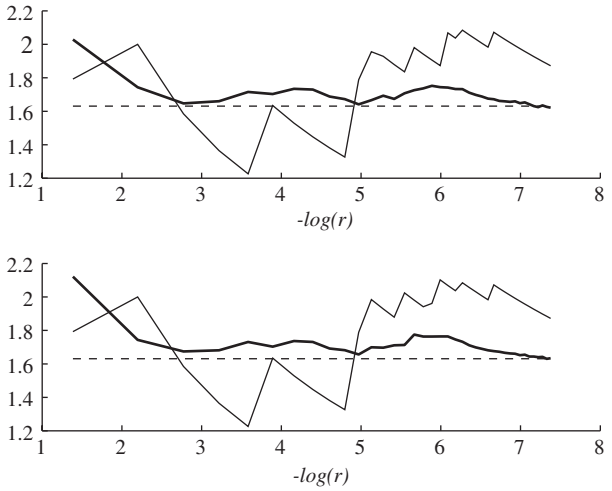

Fig. 2. On the top slanted Baker map with parameters $\lambda_1 = \lambda_2 = \frac{1}{3}$, $\mu_1 = 1$, $\mu_2 = 0$, $c_1 = 0.51$, $c_2 = -0.512211122$, whereas on the bottom non-linear slanted Baker map with the same parameters as slanted Baker and $\varepsilon = 0.01$. In both figures thin line is $-\log(\tau_r(y, y))/\log(r)$, thick line is $\langle R_r(y)\rangle$ with a mean over 50 different initial points (x_k), and horizontal dashed line represents the theoretical dimension $1 + s$ obtained from Ref. [19].

in the following figures. Fig. 1 presents the case of the Baker Map for two sets of parameters; in Figs. 2 and 3 we present similar results for the slanted Baker map and its non-linear perturbation, for two sets of parameters.

2.2. Hénon map

The *Hénon map* is a diffeomorphism of the plane $\mathbb{R}^2$ depending on two parameters, it may be expressed as

$$T^{hen}(x, y) = (y - ax^2 + 1, bx). \tag{4}$$

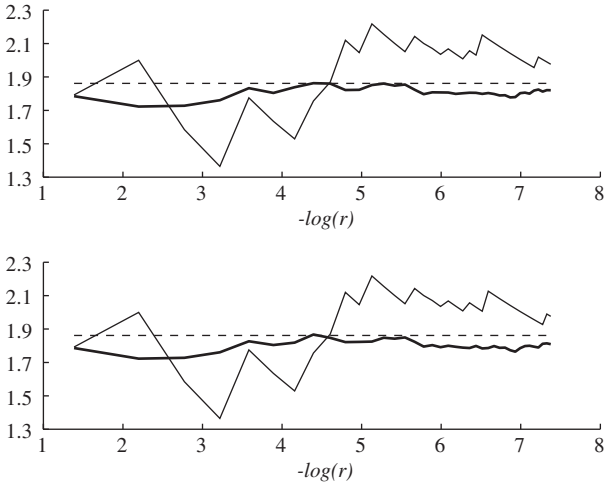

Fig. 3. On the top slanted Baker map with parameters $\lambda_1 = \lambda_2 = \frac{1}{\sqrt{5}}$, $\mu_1 = \frac{1}{6}$, $\mu_2 = 0$, $c_1 = 0.3$, $c_2 = -0.512211122$, whereas on the bottom non-linear slanted Baker map with same parameters as slanted Baker map and $\varepsilon = 0.01$. Thin line is $-\log(\tau_r(y, y))/\log(r)$, thick line is $\langle R_r(y)\rangle$ with the mean over 50 different initial points (x_k), and horizontal dashed line represents the theoretical dimension $1 + s$ obtained from Ref. [19].

In literature a second map, still called Hénon map, is given

$$T_1^{hen}(x, y) = (-b'y - x^2 + a', x), \tag{5}$$

when $a' = a$ and $b' = -b$, using the rescaling: $I_{\alpha,\beta}(x, y) = (\alpha x, \beta y)$, with $\alpha = a^{-1}$ and $\beta = b/a$, we can conjugate the two maps, i.e., $T^{hen} \circ I_{\alpha,\beta} = I_{\alpha,\beta} \circ T_1^{hen}$, thus mapping orbits onto orbits.

The attractor of the Hénon map is more involved than the one of the Baker-like maps, and moreover, it has a less uniform structure. Few rigorous results are known about it, some of them emphasise its complicate behaviour.

Hénon's attractor and its dimension have been intensively studied by means of several numerical methods (e.g., Refs. [20–22]). For parameters $a = 1.4$, $b = 0.3$, different experiments, based on independent methods (for instance, the box counting algorithm that we also used), give an estimated value of the dimension close to 1.27. This value is in good agreement with the one obtained using recurrence and waiting time indicators, see Fig. 4.

We also studied the Hénon map for a different set of parameters: $a = 1.2$, $b = 0.2$. These parameters have been chosen to approach the Devaney–Nitecky horseshoe locus where it is proved [23] that the dimension of the most natural invariant measure is strictly smaller than the dimension of its support. Since the box counting method depends on the support of the measure whereas the recurrence waiting time based methods do not, we tried to verify this phenomenon with our experiments. Comparing Figs. 4 and 5, we can see two different behaviours (in the latter one the waiting time seems to approach very slowly from below the box counting dimension), but this numerical result does not allow us to completely answer to this question.

2.3. Translation on S^1 by a well-approximable angle

Let us consider the *translation map* on the unit circle S^1: $T(x) = x + \theta \pmod 1$. We choose the angle θ to be irrational and "well approximable by rational numbers", for example some Liouville number. In this case, it is known that the quantitative recurrence and waiting times fail to give a good approximation for the dimension. The physical measure here is the Lebesgue measure and the dimension is 1 at each point. It is also known [12,15] that in this case the lower recurrence rate will be strictly smaller than 1, whereas the upper recurrence rate tends to 1. Furthermore the upper waiting time rate will be strictly larger than 1, whereas the lower one is 1. This is shown in Fig. 6, where one can see both indicators oscillating, according to theoretical predictions.

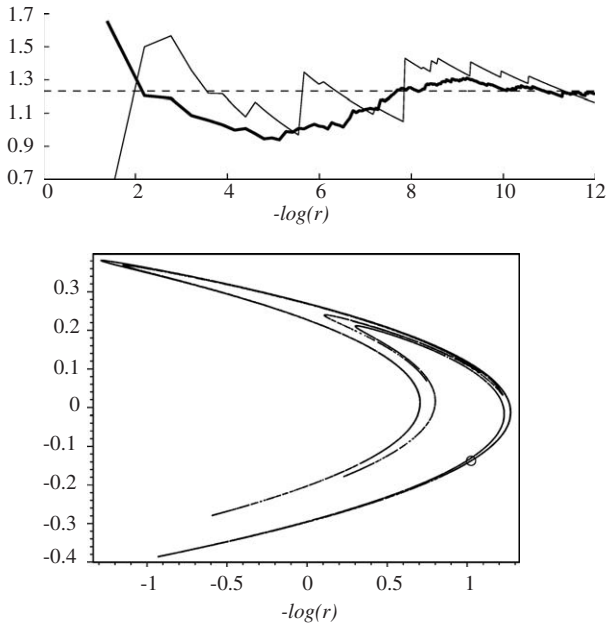

Fig. 4. Hénon map with parameters: $a = 1.4$, $b = 0.3$. On the top recurrence vs dimension, thin line is $-\log(\tau_r(y, y))/\log(r)$, thick line is $\langle R_r(y)\rangle$ with the mean over 200 different initial points x_k, and horizontal dashed line is the dimension computed using box counting method. On the bottom the Hénon attractor, the circle denotes the point where local dimension has been evaluated.

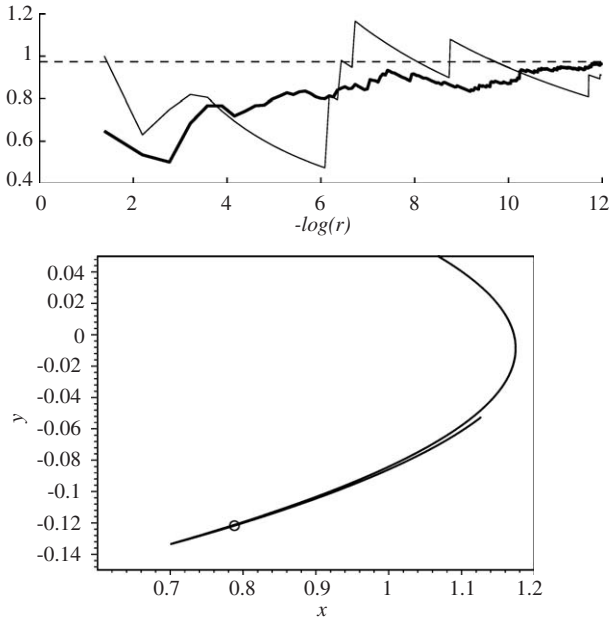

Fig. 5. Hénon map with parameters $a = 1.2$, $b = 0.2$. On the top recurrence vs dimension, thin line is $-\log(\tau_r(y, y))/\log(r)$, thick line is $\langle R_r(y)\rangle$ with the mean over 200 different initial points x_k, and horizontal dashed line is the dimension computed using box counting method. On the bottom part of the Hénon attractor, the circle denotes the point where local dimension has been evaluated.

2.4. Logistic map at Feigembaum point

The *logistic map* is the map of the interval $[0, 1]$ into itself, given by $T(x) = \mu x(1 - x)$. We considered this map with parameter $\mu_\infty = 3.569945671870944901842005151 38$, a value very close to the so called *Feigembaum*

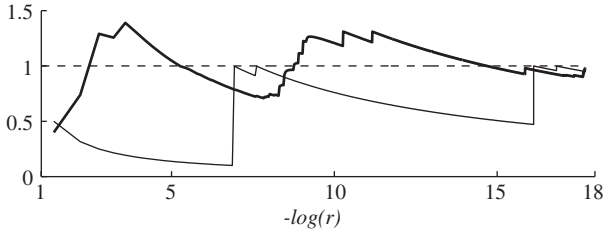

Fig. 6. Translation by the angle $\theta = 0.50050000005000000000005$. Thin line is $-\log(\tau_r(y,y))/\log(r)$, thick line is $\langle R_r(y)\rangle$ with the mean over 10 different initial points x_k, and the horizontal dashed line with ordinate 1 is the theoretical dimension.

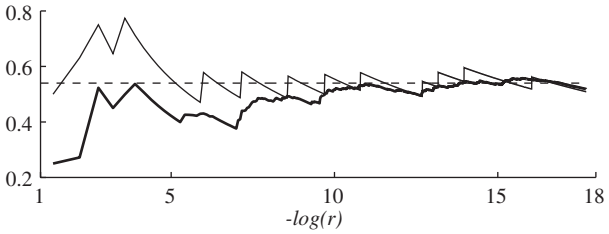

Fig. 7. Logistic map with parameter $\mu_\infty = 3.569945671870944901842005151 38$, very close to Feigembaum point. Thin line is $-\log(\tau_r(y,y))/\log(r)$, thick line is $\langle R_r(y)\rangle$ with the mean over 20 different initial points x_k and the horizontal dashed line is the dimension computed by box counting method.

point. This map is not hyperbolic, indeed it has zero entropy and Lyapunov exponent. Its attractor is a Cantor set (see Ref. [24] or Ref. [25] for a description). We estimated the dimension of the attractor by box counting and also by recurrence and waiting times. The results are reported in Fig. 7.

3. Conclusions and perspectives

As already pointed out, the recurrence has been considered to be an interesting tool to estimate local dimension of attractors, however, it has a main drawback in the presence of large fluctuations and slow convergence rate (as we emphasised in our previous numerical experiments). We believe that, the use of waiting times, due to this faster convergence can be a more interesting tool. Moreover, the waiting times shares with recurrence the very interesting property of being independent of the support of the measure, depending on the invariant measure itself. This remark opens the possibility of using such indicators for the study of systems where the properties of the measure are different from the properties of its support (for instance some Hénon maps, see Ref. [23]). This kind of questions are very interesting and they demand very subtle analysis and the need of further explorations.

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